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Asymptotic Behaviour of Bernoulli, Euler, and Generalized Bernoulli Polynomials

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1. INTRODUCTION

The question of the location of real zeros of the Bernoulli polynomials $B_n(x)$ has been subject to thorough investigation by several authors. Lense [7], Lehmer [6], Inkeri [3], Ostrowski [9], and Leeming [5] showed that, essentially, each point $(2j+1)/4$ (for integers j) on the real axis is an accumulation point of real zeros of polynomials $B_{2k}(x)$, while each $j/2$ is an accumulation point of zeros of the $B_{2k+1}(x)$. They gave very exact estimates for the distance of the zeros from the corresponding accumulation points.

Similarly in [1], the present author studied the zeros of generalized Bernoulli polynomials $B_\chi^n(z)$, where χ is a primitive quadratic character with conductor f. He showed that $f/4$ (resp. $f/2$) are accumulation points of the $B_\chi^n(z)$.

These and other properties lead to the observation that the Bernoulli polymonials $B_n(z)$, normalized in a certain way, behave like $\cos(2\pi z)$ or $\sin(2\pi z)$, and that the normalized generalized Bernoulli polynomials behave like $\cos(2\pi z/f)$ or $\sin(2\pi z/f)$. It is the aim of this note to make these statements more precise. In particular, it is shown that the polynomials in question can be approximated by the truncated MacLaurin series for sine and cosine; explicit error bounds will be given. The proofs are based on Euler's formula connecting Bernoulli numbers with the Riemann zeta function at even integer arguments, and on related generalized formulas.

An important consequence of our main results is the fact that the sequences of polynomials under consideration converge uniformly on compact subsets of $\mathbb{C}$ to the sine or cosine functions. This can be derived by different methods, using Cauchy's integral formula and complex contour

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integration, as was pointed out to the author by H. Delange. In fact, Darboux's method on the generating function (see, e.g., [11, p. 206]) is a convenient tool to derive this and other asymptotic results. This method would also allow us to find complete asymptotic expansions and explicit error bounds; this remark is due to one of the referees.

The results in this paper can be applied to the study of complex zeros of Bernoulli and generalized Bernoulli polynomials. They indicate that we can expect a distribution of zeros similar to that of the sections of the sine and cosine functions, and they can be used to find upper bounds for the size of zero-free regions. This will be discussed in a forthcoming paper.

2. Bernoulli Polynomials

The Bernoulli polynomials $B_n(z)$ can be defined by

$$B_n(z)=\sum_{j=0}^{n}\binom{n}{j}B_j z^{n-j}, \tag{1}$$

where the B_j are the Bernoulli numbers, given by

$$t/(e^t-1)=\sum_{j=0}^{\infty} B_j t^j/j! \qquad (|t|<2\pi).$$

We denote the sections of the cosine and sine functions by

$$T_{2k}(z)=\sum_{j=0}^{k}(-1)^j z^{2j}/(2j)!,$$
$$T_{2k+1}(z)=\sum_{j=0}^{k}(-1)^j z^{2j+1}/(2j+1)!.$$

For real x, let $[x]$ denote the integral part of x.

THEOREM 1. *For all* $z\in\mathbb{C}$, $n\geqslant 2$, *we have* (*with* $k=[n/2]$)

$$\left|(-1)^k\frac{(2\pi)^n}{2n!}B_n\left(z+\frac{1}{2}\right)-T_n(2\pi z)\right|<2^{-n}\exp(4\pi\,|z|).$$

We know that $T_n(2\pi z)$ is uniformly convergent on a compact subset to $\cos(2\pi z)$ if n is even, and to $\sin(2\pi z)$ if n is odd. Hence we have, after replacing $z+\frac{1}{2}$ by z, COROLLARY 1. *The following sequences converge uniformly on compact subsets of* $\mathbb{C}$:

$$(-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}(z)\to\cos(2\pi z),$$

$$(-1)^{k-1}\frac{(2\pi)^{2k+1}}{2(2k+1)!}B_{2k+1}(z)\to\sin(2\pi z).$$

Proof of Theorem 1. If we expand $B_n(z+\frac{1}{2})$ in a Taylor series, noting that $B'_n(x)=nB_{n-1}(x)$ and

$$B_{2j}(\tfrac{1}{2})=(2^{1-2j}-1)\,B_{2j},\qquad B_{2j-1}(\tfrac{1}{2})=0\qquad (j\geqslant 1)$$

(see, e.g., [10, p. 13]), we find

$$B_n(z+\tfrac{1}{2})=\sum_{j=0}^{k}\binom{n}{2j}(2^{1-2j}-1)\,B_{2j}z^{n-2j}.$$

With the well-known formula of Euler (see, e.g., [10, p. 16])

$$B_{2j}=(-1)^{j-1}\,2(2j)!\,(2\pi)^{-2j}\,\zeta(2j)$$

(for $j\geqslant 1$), where $\zeta(z)$ is the Riemann zeta function, we get

$$\frac{(2\pi)^n}{2n!}B_n\left(z+\frac{1}{2}\right)=\frac{(2\pi z)^n}{2n!}+\sum_{j=1}^{k}(-1)^{j-1}\frac{(2\pi z)^{n-2j}}{(n-2j)!}(2^{1-2j}-1)\,\zeta(2j).$$

Hence

$$\frac{(2\pi)^n}{2n!}B_n\left(z+\frac{1}{2}\right)-(-1)^k\,T_n(2\pi z)=A_n(z)-\frac{(2\pi z)^n}{2n!},\tag{2}$$

where

$$A_n(z):=\sum_{j=1}^{k}(-1)^{j-1}\frac{(2\pi z)^{n-2j}}{(n-2j)!}\sum_{m=2}^{\infty}(-1)^m\,m^{-2j}.$$

Because

$$0<\sum_{m=2}^{\infty}(-1)^m\,m^{-2j}<2^{-2j}$$

we have

$$\left|A_n(z) - \frac{(2\pi z)^n}{2n!}\right| \leqslant 2^{-n} \sum_{j=1}^{k} \frac{(4\pi |z|)^{n-2j}}{(n-2j)!} + 2^{-n} \frac{(4\pi |z|)^n}{2n!}$$
$$\leqslant 2^{-n} \exp(4\pi |z|).$$

This, together with (2), proves Theorem 1.

3. Generalized Bernoulli Polynomials

The Bernoulli numbers and polynomials have been generalized in various directions. The following generalization is of particular interest in number theory.

A residue class character χ with modulus m is a mapping $\chi\colon \mathbb{Z} \to \mathbb{C}$ such that (i) $\chi(a) = 0$ if and only if $\gcd(a, m) \neq 1$; (ii) if $a \equiv b \pmod m$ then $\chi(a) = \chi(b)$; (iii) $\chi(ab) = \chi(a)\,\chi(b)$. It follows that $|\chi(a)| = 1$ for every integer a, $\gcd(a, m) = 1$. If m divides m' and χ is a character with modulus m, it induces a character χ' with modulus m' as follows: if $\gcd(a, m') \neq 1$ then $\chi'(a) = 0$; if $\gcd(a, m') = 1$ then $\chi'(a) = \chi(a)$. A character χ with modulus m is called primitive if it is not induced from any character with modulus less than m. In this case m is called the conductor of χ and is denoted by f_χ or f. The character $\bar{\chi}$ is defined by $\bar{\chi}(a) = \overline{\chi(a)}$ (the complex conjugate) for all a. If $(\chi(a))^2 = 1$ if $\gcd(a, f) = 1$, then χ is called quadratic.

Let χ be a primitive residue class character with conductor f; the generalized Bernoulli numbers B_χ^n are defined by

$$\sum_{a=1}^{f} \frac{\chi(a)\, te^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_\chi^n t^n/n!. \tag{3}$$

If $f = 1$, i.e., if χ is the principal character ($\chi(a) = 1$ for all a), we get the Bernoulli numbers B_n, except for $B_\chi^1 = \frac{1}{2} = -B_1$. As a second example we regard the quadratic character with $f = 4$ defined by $\chi(1) = 1$, $\chi(3) = -1$, $\chi(0) = \chi(2) = 0$. The left-hand side of (3) becomes $-t/(e^t + e^{-t})$. Comparing this with the generating function of the Euler numbers (see, e.g., [2]) $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61, \ldots,$ and $E_{2n+1} = 0$, we find that $2B_\chi^n = -nE_{n-1}$ $(n \geqslant 1)$.

We note that in general the B_χ^n are algebraic numbers, belonging to the smallest algebraic number field containing all $\chi(a)$. We define $\delta = \delta_\chi$ by $\delta_\chi = 0$ or 1 according to whether χ is even (i.e., $\chi(-1) = 1$) or odd (i.e., $\chi(-1) = -1$), and we note the following important formulas (see, e.g., [4, Chap. 2]):

$$B_\chi^m = 0 \qquad \text{if} \quad m \not\equiv \delta \pmod 2; \tag{4}$$

if $m \equiv \delta \pmod 2$, $m \geqslant 1$, then

$$B_\chi^m = (-1)^{1+(m-\delta)/2} \frac{2m!\, i^\delta f^m}{(2\pi)^m\, G(\bar{\chi})} L(m, \bar{\chi}), \tag{5}$$

with Dirichlet's L-function

$$L(m, \chi) = \sum_{\nu=1}^{\infty} \chi(\nu)\, \nu^{-m}$$

and the Gaussian sum

$$G(\chi) = \sum_{a=1}^{f} \chi(a) \exp(2\pi i a/f).$$

The generalized Bernoulli polynomials are now defined by

$$B_\chi^m(z) := \sum_{j=0}^{n} \binom{n}{j} B_\chi^j z^{n-j}. \tag{6}$$

Analogous to Theorem 1 we have

THEOREM 2. *If $f > 1$ and $k := [(n+\delta)/2]$ then we have for all $z \in \mathbb{C}$, $n \geqslant 2$,*

$$\left| (-1)^k \frac{(2\pi/f)^n}{2n!} i^\delta G(\bar{\chi})\, B_\chi^n(z) + T_{n+\delta-2}\left(\frac{2\pi z}{f}\right) \right| \leqslant 2^{-n+1}\pi \sqrt{f} \exp\left(\frac{4\pi\, |z|}{f}\right).$$

As a direct consequence we obtain

COROLLARY 2. *The following subsequences are uniformly convergent on compact subsets of $\mathbb{C}$ as $n \to \infty$:*

$$(-1)^{k-1} \frac{(2\pi/f)^n}{2n!} i^\delta G(\bar{\chi})\, B_\chi^n(z) \to \begin{cases} \cos\left(\dfrac{2\pi z}{f}\right) & \text{for } n \equiv \delta \pmod 2 \\ \sin\left(\dfrac{2\pi z}{f}\right) & \text{for } n \not\equiv \delta \pmod 2. \end{cases}$$

Proof of Theorem 2. With $k = [(n+\delta)/2]$, (6), (4), and (5) yield

$$B_\chi^n(z) = \sum_{j=1}^{k} \binom{n}{2j-\delta} z^{n-2j+\delta}(-1)^{j+1-\delta} \frac{2(2j-\delta)!\, f^{2j-\delta} i^\delta}{(2\pi)^{2j-\delta}\, G(\bar{\chi})} L(2j-\delta, \bar{\chi}),$$

or

$$\frac{(2\pi/f)^n}{2n!} i^\delta G(\bar{\chi}) B_\chi^n(z) = \sum_{j=1}^{k} (-1)^{j+1} \frac{(2\pi z/f)^{n-2j+\delta}}{(n-2j+\delta)!} L(2j-\delta, \bar{\chi})$$
$$= \sum_{j=1}^{k} (-1)^{j+1} \frac{(2\pi z/f)^{n-2j+\delta}}{(n-2j+\delta)!} + C_n(z),$$

where

$$C_n(z) = \sum_{j=1}^{k} (-1)^{j-1} \frac{(2\pi z/f)^{n-2j+\delta}}{(n-2j+\delta)!} \sum_{v=2}^{\infty} \bar{\chi}(v)\, v^{-2j+\delta},$$

and hence

$$\frac{(2\pi/f)^n}{2n!} i^\delta G(\bar{\chi}) B_\chi^n(z) = (-1)^{k-1} T_{n+\delta-2}\left(\frac{2\pi z}{f}\right) + C_n(z). \tag{7}$$

To estimate $C_n(z)$, we first observe that for $a \geqslant 2$,

$$\left| \sum_{v=2}^{\infty} \bar{\chi}(v)\, v^{-a} \right| \leqslant \sum_{v=2}^{\infty} v^{-a} < 3 \cdot 2^{-a}. \tag{8}$$

In the case $a = 1$ we could apply one of the well-known but deep results on the size of $L(1, \chi)$; however, for our purposes it suffices to do the following. By (5) we have, with $\delta = 1$,

$$\sum_{v=1}^{\infty} \chi(v)\, v^{-1} = -\frac{\pi G(\chi)}{if} B_1^\chi,$$

and also (see, e.g., [4, p. 14])

$$B_1^\chi = f^{-1} \sum_{a=1}^{f} \chi(a)\, a, \qquad |B_1^\chi| < f^{-1} f^2 = f.$$

Therefore, if we use the well-known fact $|G(\bar{\chi})| = \sqrt{f}$, we get

$$\left| \sum_{v=2}^{\infty} \bar{\chi}(v)\, v^{-1} \right| < 2\pi \sqrt{f}\, 2^{-1}.$$

We combine this with (8); so for $j \geqslant 1$ and $\delta = 0$ or 1 we have

$$\left| \sum_{v=2}^{\infty} \bar{\chi}(v)\, v^{-2j+\delta} \right| < 2\pi \sqrt{f}\, 2^{-2j+\delta}.$$

Hence

$$
\begin{aligned}
|C_n(z)| &\leqslant \sum_{j=1}^{k} 2\pi \sqrt{f}\, 2^{-2j+\delta}(2\pi\, |z|/f)^{n-2j+\delta}/(n-2j+\delta)! \\
&= \pi \sqrt{f}\, 2^{-n+1} \sum_{j=1}^{k} (4\pi\, |z|/f)^{n-2j+\delta}/(n-2j+\delta)! \\
&\leqslant \pi \sqrt{f}\, 2^{-n+1} \exp(4\pi\, |z|/f).
\end{aligned}
$$

Theorem 2 now follows immediately from (7).

Remark. If χ is an even character, i.e., $\delta = 0$, then it is clear from the proof that the error bound in Theorem 2 can be improved to $2^{-n} \exp(4\pi\, |/f)$. In the case $\delta = 1$, the factor $\sqrt{f}$ could be improved to the order of $\operatorname{Log} f$.

4. Euler Polynomials

As an example of Corollary 2 we consider again the character χ with $\chi(1) = 1$, $\chi(3) = -1$, $\chi(0) = \chi(2) = 0$, and $f_\chi = 4$. A known relation between generalized Bernoulli polynomials and (classical) Bernoulli polynomials (see, e.g., [4, p. 10]), gives for this particular character

$$
B_\chi^n(z) = 4^{n-1}\left[B_n\left(\frac{z+1}{4}\right) - B_n\left(\frac{z+3}{4}\right)\right].
$$

Furthermore, the Euler polynomials $E_n(z)$ are related to the Bernoulli polynomials by

$$
nE_{n-1}(z) = 2^n\left[B_n\left(\frac{z+1}{2}\right) - B_n\left(\frac{z}{2}\right)\right]
$$

(see, e.g., [2]). This gives

$$
B_\chi^n(z) = -n2^{n-2}E_{n-1}\left(\frac{z+1}{2}\right).
$$

Now we apply Corollary 2, using the fact that $G(\bar{\chi}) = G(\chi) = i\sqrt{f}$ for odd quadratic characters. If we change the index n to $n+1$ and the variable z to $2z-1$, taking into account $\cos(\pi z - \pi/2) = \sin(\pi z)$, $\sin(\pi z - \pi/2) = -\cos(\pi z)$, we get

COROLLARY 3. *The following sequences converge uniformly on compact subsets of* $\mathbb{C}$:

$$(-1)^k \frac{\pi^{2k+1}}{4(2k)!} E_{2k}(z) \to \sin(\pi z)$$

$$(-1)^{k+1} \frac{\pi^{2k+2}}{4(2k+1)!} E_{2k+1}(z) \to \cos(\pi z).$$

We can now apply the theorem of Hurwitz on convergent series of analytic functions (see, e.g., Marden [8, p. 4]). Let t be any integer and $\varepsilon > 0$ a sufficiently small real number. Then

1. the circular region around the point t with radius ε contains exactly one simple zero of $E_{2k}(z)$ for all $k \geqslant m(t, \varepsilon)$;

2. the circular region around $(2t+1)/2$ with radius ε contains exactly one simple zero of $E_{2k+1}(z)$ for all $k \geqslant n(t, \varepsilon)$.

This is consistent with and supplements the results of Howard [2] on the real zeros of Euler polynomials.

In the same way, Corollaries 1 and 2 lead to the corresponding statements about the zeros of Bernoulli and generalized Bernoulli polynomials. This may serve as an explanation for the nature of the results mentioned in the Introduction.

5. POLYNOMIALS RELATED TO $B_n(z)$, $B_\chi^n(z)$

We know that $B_{2k}(0) = B_{2k} \neq 0$ for $k \geqslant 0$, and $B_\chi^n(0) = B_\chi^n \neq 0$ if χ is a primitive character and $n \equiv \delta_\chi \pmod 2$. In these cases it is of interest to study the polynomials

$$P_{2k}(z) := B_{2k}(z) - B_{2k}$$

$$P_\chi^n(z) := B_\chi^n(z) - B_\chi^n.$$

COROLLARY 4. *The following converge uniformly on compact subsets*:

(a) $$(-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} P_{2k}(z) \to \cos(2\pi z) - 1;$$

(b) $$(-1)^{k-1+\delta} \frac{(2\pi/f)^{2k+\delta}}{2(2k+\delta)!} i^\delta G(\bar{\chi})\, P_\chi^{2k+\delta}(z) \to \cos(2\pi z/f) - 1.$$

Proof. (a) In the proof of Theorem 1, the equation before (2) becomes (with $n=2k$)

$$\frac{(2\pi)^n}{2n!} P_n\left(z+\frac{1}{2}\right)=\sum_{j=1}^{k-1} (-1)^{j-1} \frac{(2\pi z)^{n-2j}}{(n-2j)!} \zeta(2j)(2^{1-2j}-1)+\frac{(2\pi z)^n}{2n!},$$

or

$$\frac{(2\pi)^n}{2n!} P_n\left(z+\frac{1}{2}\right)-(-1)^k (T_n(2\pi z)-1)=A_n^*(z)-\frac{(2\pi z)^n}{2n!}, \qquad (2')$$

where $A_n^*(z)$ differs from $A_n(z)$ only in that the first summation goes up to $k-1$ (instead of k). The result now follows from (2′), just as before.

(b) In exact analogy to (a), we see in the proof of Theorem 2 that (7) becomes (with $n=2k+\delta$)

$$\frac{(2\pi/f)^n}{2n!} i^\delta G(\bar{\chi}) P_\chi^n(z)=(-1)^{k-1}\left(T_{n+\delta-2}\left(\frac{2\pi z}{f}\right)-1\right)+C_n^*(z), \quad (7')$$

where $C_n^*(z)$ is defined like $C_n(z)$, except that the first summation goes only up to $k-1$. The result follows from (7′) in the same manner as before.

Finally, we can make the following remark. Let $F(z)$ be any entire function. If we take $B_n(z)+F(z)$ instead of $B_n(z)$ in Corollary 1, $B_\chi^n(z)+F(z)$ instead of $B_\chi^n(z)$ in Corollary 2, $P_{2k}(z)+F(z)$ and $P_\chi^{2k+\delta}(z)+F(z)$ instead of $P_{2k}(z)$ and $P_\chi^{2k+\delta}(z)$ in Corollary 4, then it is easy to see that the new sequences converge like the original ones.

Indeed, consider the proof of Theorem 1. On the right-hand side of (2) there would appear the additional term $F(z)(2\pi)^n/2n!$, and therefore the right-hand side of the inequality in Theorem 1 would become

$$2^{-n}\exp(4\pi\,|z|)+|F(z)|\,(2\pi)^n/2n!$$

This tends uniformly to zero on any compact, which proves the claim for $B_n(z)+F(z)$. The other cases can be verified in the same way.

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